

# Binomial Theorem and Mathematical Induction

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Assignment (Basic and Advance Level)

Answer Sheet of Assignment



Blaise Pascal

*The ancient Indian mathematicians knew the coefficient in the expansion of  $(x + y)^n$ ,  $0 \leq n \leq 7$ . The arrangement of these coefficient was in the form of a diagram called Meru-Prastara, provided by Pingla in his book Chhanda-shastra (200 B.C.). The term binomial coefficients was first introduced by the German mathematician. Michael Stipel (1486-1567 A.D.)*

*The arithmetic triangle popularly known as pascal triangle was constructed by the French mathematician Blaise Pascal (1623-1662 A.D.) He used the triangle to derive coefficients of a binomial expansion. It was printed in 1665 A.D. The present form of the binomial theorem for integral values of  $n$  appeared in Trate du triangle arithmetic written by Pascal and published posthumously in 1665 A.D. The generalization of the binomial theorem for negative integral and rational exponents is due to Sir Isaac Newton (1642-1727 A.D) in the same year 1665.*

## 6.1 Binomial Theorem

### 6.1.1 Binomial Expression

An algebraic expression consisting of two terms with +ve or - ve sign between them is called a binomial expression.

For example :  $(a + b), (2x - 3y), \left(\frac{p}{x^2} - \frac{q}{x^4}\right), \left(\frac{1}{x} + \frac{4}{y^3}\right)$  etc.

### 6.1.2 Binomial Theorem for Positive Integral Index

The rule by which any power of binomial can be expanded is called the binomial theorem.

If  $n$  is a positive integer and  $x, y \in C$  then

$$(x + y)^n = {}^n C_0 x^{n-0} y^0 + {}^n C_1 x^{n-1} y^1 + {}^n C_2 x^{n-2} y^2 + \dots + {}^n C_r x^{n-r} y^r + \dots + {}^n C_{n-1} x y^{n-1} + {}^n C_n x^0 y^n$$

$$\text{i.e., } (x + y)^n = \sum_{r=0}^n {}^n C_r x^{n-r} y^r \quad \dots \text{(i)}$$

Here  ${}^n C_0, {}^n C_1, {}^n C_2, \dots, {}^n C_n$  are called binomial coefficients and  ${}^n C_r = \frac{n!}{r!(n-r)!}$  for  $0 \leq r \leq n$ .

#### Important Tips

- ☞ The number of terms in the expansion of  $(x + y)^n$  are  $(n + 1)$ .
- ☞ The expansion contains decreasing power of  $x$  and increasing power of  $y$ . The sum of the powers of  $x$  and  $y$  in each term is equal to  $n$ .
- ☞ The binomial coefficients  ${}^n C_0, {}^n C_1, {}^n C_2, \dots$  equidistant from beginning and end are equal i.e.,  ${}^n C_r = {}^n C_{n-r}$ .
- ☞  $(x + y)^n = \text{Sum of odd terms} + \text{sum of even terms}$ .

### 6.1.3 Some Important Expansions

(1) Replacing  $y$  by  $-y$  in (i), we get,

$$(x - y)^n = {}^n C_0 x^{n-0} y^0 - {}^n C_1 x^{n-1} y^1 + {}^n C_2 x^{n-2} y^2 - \dots + (-1)^r {}^n C_r x^{n-r} y^r + \dots + (-1)^n {}^n C_n x^0 y^n$$

$$\text{i.e., } (x - y)^n = \sum_{r=0}^n (-1)^r {}^n C_r x^{n-r} y^r \quad \dots \text{(ii)}$$

The terms in the expansion of  $(x - y)^n$  are alternatively positive and negative, the last term is positive or negative according as  $n$  is even or odd.

(2) Replacing  $x$  by 1 and  $y$  by  $x$  in equation (i) we get,

$$(1 + x)^n = {}^n C_0 x^0 + {}^n C_1 x^1 + {}^n C_2 x^2 + \dots + {}^n C_r x^r + \dots + {}^n C_n x^n \text{ i.e., } (1 + x)^n = \sum_{r=0}^n {}^n C_r x^r$$

This is expansion of  $(1 + x)^n$  in ascending power of  $x$ .

(3) Replacing  $x$  by  $1$  and  $y$  by  $-x$  in (i) we get,

$$(1-x)^n = {}^n C_0 x^0 - {}^n C_1 x^1 + {}^n C_2 x^2 - \dots + (-1)^r {}^n C_r x^r + \dots + (-1)^n {}^n C_n x^n \text{ i.e., } (1-x)^n = \sum_{r=0}^n (-1)^r {}^n C_r x^r$$

(4)  $(x+y)^n + (x-y)^n = 2[{}^n C_0 x^n y^0 + {}^n C_2 x^{n-2} y^2 + {}^n C_4 x^{n-4} y^4 + \dots]$  and

$$(x+y)^n - (x-y)^n = 2[{}^n C_1 x^{n-1} y^1 + {}^n C_3 x^{n-3} y^3 + {}^n C_5 x^{n-5} y^5 + \dots]$$

(5) The coefficient of  $(r+1)^{\text{th}}$  term in the expansion of  $(1+x)^n$  is  ${}^n C_r$ .

(6) The coefficient of  $x^r$  in the expansion of  $(1+x)^n$  is  ${}^n C_r$ .

**Note** :  $\square$  If  $n$  is odd, then  $(x+y)^n + (x-y)^n$  and  $(x+y)^n - (x-y)^n$ , both have the same number of terms equal to  $\left(\frac{n+1}{2}\right)$ .

$\square$  If  $n$  is even, then  $(x+y)^n + (x-y)^n$  has  $\left(\frac{n}{2} + 1\right)$  terms and  $(x+y)^n - (x-y)^n$  has  $\frac{n}{2}$  terms.

**Example: 1**  $x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5 =$   
 (a)  $(x+a)^5$  (b)  $(3x+a)^5$  (c)  $(x+2a)^5$  (d)  $(x+2a)^3$

**Solution:** (c) Conversely  $(x+y)^n = {}^n C_0 + {}^n C_1 x^{n-1} y^1 + {}^n C_2 x^{n-2} y^2 + \dots + {}^n C_n x^0 y^n$   
 $(x+2a)^5 = {}^5 C_0 x^5 + {}^5 C_1 x^4 (2a)^1 + {}^5 C_2 x^3 (2a)^2 + {}^5 C_3 x^2 (2a)^3 + {}^5 C_4 x^1 (2a)^4 + {}^5 C_5 x^0 (2a)^5$   
 $= x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5 .$

**Example: 2** The value of  $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$  will be  
 (a)  $-198$  (b)  $198$  (c)  $98$  (d)  $-99$

**Solution:** (b) We know that,  $(x+y)^n + (x-y)^n = 2[{}^n C_0 x^n + {}^n C_2 x^{n-2} y^2 + {}^n C_4 x^{n-4} y^4 + \dots]$   
 $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2[({}^6 C_0 (\sqrt{2})^6 + {}^6 C_2 (\sqrt{2})^4 (1)^2 + {}^6 C_4 (\sqrt{2})^2 (1)^4 + {}^6 C_6 (\sqrt{2})^0 (1)^6] = 2[8 + 15 \times 4 + 30 + 1] = 198$

**Example: 3** The larger of  $99^{50} + 100^{50}$  and  $101^{50}$  is [IIT 1980]  
 (a)  $99^{50} + 100^{50}$  (b) Both are equal (c)  $101^{50}$  (d) None of these

**Solution:** (c) We have,  $101^{50} = (100+1)^{50} = 100^{50} + 50 \cdot 100^{49} + \frac{50 \cdot 49}{2 \cdot 1} 100^{48} + \dots$  .....(i)  
 and  $99^{50} = (100-1)^{50} = 100^{50} - 50 \cdot 100^{49} + \frac{50 \cdot 49}{2 \cdot 1} 100^{48} - \dots$  ..... (ii)  
 Subtracting,  $101^{50} - 99^{50} = 100^{50} + 2 \cdot \frac{50 \cdot 49 \cdot 48}{3 \cdot 2 \cdot 1} 100^{47} + \dots > 100^{50}$ . Hence  $101^{50} > 100^{50} + 99^{50}$ .

**Example: 4** Sum of odd terms is  $A$  and sum of even terms is  $B$  in the expansion of  $(x+a)^n$ , then

(a)  $AB = \frac{1}{4}(x-a)^{2n} - (x+a)^{2n}$  (b)  $2AB = (x+a)^{2n} - (x-a)^{2n}$   
 (c)  $4AB = (x+a)^{2n} - (x-a)^{2n}$  (d) None of these

**Solution:** (c)  $(x+a)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1} a^1 + {}^n C_2 x^{n-2} a^2 + \dots + {}^n C_n x^{n-n} a^n = (x^n + {}^n C_2 x^{n-2} a^2 + \dots) + ({}^n C_1 x^{n-1} a^1 + {}^n C_3 x^{n-3} a^3 + \dots) = A + B$  .....(i)  
 Similarly,  $(x-a)^n = A - B$  .....(ii)

From (i) and (ii), we get  $4AB = (x+a)^{2n} - (x-a)^{2n}$

**Trick:** Put  $n=1$  in  $(x+a)^n$ . Then,  $x+a = A+B$ . Comparing both sides  $A=x, B=a$ .

Option (c) L.H.S.  $4AB = 4xa$ , R.H.S.  $(x+a)^2 - (x-a)^2 = 4ax$ . i.e., L.H.S. = R.H.S

### 6.1.4 General Term

$$(x+y)^n = {}^n C_0 x^n y^0 + {}^n C_1 x^{n-1} y^1 + {}^n C_2 x^{n-2} y^2 + \dots + {}^n C_r x^{n-r} y^r + \dots + {}^n C_n x^0 y^n$$

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The first term =  ${}^n C_0 x^n y^0$

The second term =  ${}^n C_1 x^{n-1} y^1$ . The third term =  ${}^n C_2 x^{n-2} y^2$  and so on

The term  ${}^n C_r x^{n-r} y^r$  is the  $(r+1)^{\text{th}}$  term from beginning in the expansion of  $(x+y)^n$ .

Let  $T_{r+1}$  denote the  $(r+1)^{\text{th}}$  term  $\therefore T_{r+1} = {}^n C_r x^{n-r} y^r$

This is called general term, because by giving different values to  $r$ , we can determine all terms of the expansion.

In the binomial expansion of  $(x-y)^n$ ,  $T_{r+1} = (-1)^r {}^n C_r x^{n-r} y^r$

In the binomial expansion of  $(1+x)^n$ ,  $T_{r+1} = {}^n C_r x^r$

In the binomial expansion of  $(1-x)^n$ ,  $T_{r+1} = (-1)^r {}^n C_r x^r$

**Note:**  $\square$  In the binomial expansion of  $(x+y)^n$ , the  $p^{\text{th}}$  term from the end is  $(n-p+2)^{\text{th}}$  term from beginning.

### Important Tips

$\curvearrowright$  In the expansion of  $(x+y)^n$ ,  $n \in \mathbb{N}$

$$\frac{T_{r+1}}{T_r} = \left( \frac{n-r+1}{r} \right) \frac{y}{x}$$

$\curvearrowright$  The coefficient of  $x^{n-1}$  in the expansion of  $(x-1)(x-2)\dots(x-n) = -\frac{n(n+1)}{2}$

$\curvearrowright$  The coefficient of  $x^{n-1}$  in the expansion of  $(x+1)(x+2)\dots(x+n) = \frac{n(n+1)}{2}$

**Example: 5** If the 4<sup>th</sup> term in the expansion of  $(px + x^{-1})^m$  is 2.5 for all  $x \in \mathbb{R}$  then

- (a)  $p = 5/2, m = 3$       (b)  $p = \frac{1}{2}, m = 6$       (c)  $p = -\frac{1}{2}, m = 6$       (d) None of these

**Solution:** (b) We have  $T_4 = \frac{5}{2} \Rightarrow T_{3+1} = \frac{5}{2} \Rightarrow {}^m C_3 (px)^{m-3} \left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^m C_3 p^{m-3} x^{m-6} = \frac{5}{2}$  .....(i)

Clearly, R.H.S. of the above equality is independent of  $x$

$$\therefore m-6=0, m=6$$

Putting  $m=6$  in (i) we get  ${}^6 C_3 p^3 = \frac{5}{2} \Rightarrow p = \frac{1}{2}$ . Hence  $p = 1/2, m = 6$ .

**Example: 6** If the second, third and fourth term in the expansion of  $(x+a)^n$  are 240, 720 and 1080 respectively, then the value of  $n$  is

[Kurukshetra CEE 1991; DCE 1995, 2001]

- (a) 15      (b) 20      (c) 10      (d) 5

**Solution:** (d) It is given that  $T_2 = 240, T_3 = 720, T_4 = 1080$

$$\text{Now, } T_2 = 240 \Rightarrow T_2 = {}^n C_1 x^{n-1} a^1 = 240 \quad \dots\text{(i)} \quad \text{and} \quad T_3 = 720 \Rightarrow T_3 = {}^n C_2 x^{n-2} a^2 = 720 \quad \dots\text{(ii)}$$

$$T_4 = 1080 \Rightarrow T_4 = {}^n C_3 x^{n-3} a^3 = 1080 \quad \dots\text{(iii)}$$

$$\text{To eliminate } x, \frac{T_2 \cdot T_4}{T_3^2} = \frac{240 \cdot 1080}{720 \cdot 720} = \frac{1}{2} \Rightarrow \frac{T_2}{T_3} \cdot \frac{T_4}{T_3} = \frac{1}{2}$$

$$\text{Now } \frac{T_{r+1}}{T_r} = \frac{{}^n C_r}{{}^n C_{r-1}} = \frac{n-r+1}{r}. \text{ Putting } r=3 \text{ and } 2 \text{ in above expression, we get } \frac{n-2}{3} \cdot \frac{2}{n-1} = \frac{1}{2} \Rightarrow n=5$$

**Example: 7** The 5<sup>th</sup> term from the end in the expansion of  $\left(\frac{x^3}{2} - \frac{2}{x^3}\right)^9$  is

- (a)  $63x^3$  (b)  $-\frac{252}{x^3}$  (c)  $\frac{672}{x^{18}}$  (d) None of these

**Solution:** (b) 5<sup>th</sup> term from the end =  $(9 - 5 + 2)^{\text{th}}$  term from the beginning in the expansion of  $\left(\frac{x^3}{2} - \frac{2}{x^3}\right)^9 = T_6$

$$\Rightarrow T_6 = T_{5+1} = {}^9C_5 \left(\frac{x^3}{2}\right)^4 \left(-\frac{2}{x^3}\right)^5 = -{}^9C_4 \cdot 2 \cdot \frac{1}{x^3} = -\frac{252}{x^3}.$$

**Example: 8** If  $\frac{T_2}{T_3}$  in the expansion of  $(a+b)^n$  and  $\frac{T_3}{T_4}$  in the expansion of  $(a+b)^{n+3}$  are equal, then  $n =$

[Rajasthan PET 1987, 96]

- (a) 3 (b) 4 (c) 5 (d) 6

**Solution:** (c)  $\therefore \frac{T_2}{T_3} = \frac{2}{n-2+1} \cdot \frac{b}{a} = \frac{2}{n-1} \left(\frac{b}{a}\right)$  and  $\frac{T_3}{T_4} = \frac{3}{n+3-3+1} \cdot \left(\frac{b}{a}\right) = \frac{3}{n+1} \left(\frac{b}{a}\right)$

$$\therefore \frac{T_2}{T_3} = \frac{T_3}{T_4} \quad (\text{given}) ; \therefore \frac{2}{n-1} \left(\frac{b}{a}\right) = \frac{3}{n+1} \left(\frac{b}{a}\right) \Rightarrow 2n+2 = 3n-3 \Rightarrow n=5$$

### 6.1.5 Independent Term or Constant Term

Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero.

**Condition :**  $(n-r)$  [Power of  $x$ ] +  $r$  . [Power of  $y$ ] = 0, in the expansion of  $[x+y]^n$  .

**Example: 9** The term independent of  $x$  in the expansion of  $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$  will be

[IIT 1965; BIT Ranchi 1993; Karnataka CET 2000; UPSEAT 2001]

- (a)  $\frac{3}{2}$  (b)  $\frac{5}{4}$  (c)  $\frac{5}{2}$  (d) None of these

**Solution:** (b)  $(10-r)\left(\frac{1}{2}\right) + r(-2) = 0 \Rightarrow r = 2 \therefore T_3 = {}^{10}C_2 \left(\frac{1}{3}\right)^{8/2} \left(\frac{3}{2}\right)^2 = \frac{5}{4}$

**Example: 10** The term independent of  $x$  in the expansion of  $(1+x)^n \left(1 + \frac{1}{x}\right)^n$  is [EAMCET 1989]

- (a)  $C_0^2 + 2C_1^2 + \dots + (n+1)C_n^2$  (b)  $(C_0 + C_1 + \dots + C_n)^2$  (c)  $C_0^2 + C_1^2 + \dots + C_n^2$  (d)

**Solution:** (c) We know that,  $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$

$$\left(1 + \frac{1}{x}\right)^n = {}^nC_0 + {}^nC_1 \frac{1}{x} + {}^nC_2 \frac{1}{x^2} + \dots + {}^nC_n \frac{1}{x^n}$$

Obviously, the term independent of  $x$  will be  ${}^nC_0 \cdot {}^nC_0 + {}^nC_1 \cdot {}^nC_1 + \dots + {}^nC_n \cdot {}^nC_n = C_0^2 + C_1^2 + \dots + C_n^2$

**Trick :** Put  $n=1$  in the expansion of  $(1+x)^1 \left(1 + \frac{1}{x}\right)^1 = 1+x + \frac{1}{x} + 1 = 2+x + \frac{1}{x} \dots$  (i)

We want coefficient of  $x^0$ . Comparing to equation (i). Then, we get 2 i.e., independent of  $x$ .

Option (c) :  $C_0^2 + C_1^2 + \dots + C_n^2$ ; Put  $n=1$ ; Then  ${}^1C_0^2 + {}^1C_1^2 = 1+1=2$ .

**Example: 11** The coefficient of  $x^{-7}$  in the expansion of  $\left(ax - \frac{1}{bx^2}\right)^{11}$  will be [IIT 1967; Rajasthan PET 1996]

- (a)  $\frac{462a^6}{b^5}$  (b)  $\frac{462a^5}{b^6}$  (c)  $-\frac{462a^5}{b^6}$  (d)  $-\frac{462a^6}{b^5}$

**Solution:** (b) For coefficient of  $x^{-7}$ ,  $(11-r)(1) + (-2)r = -7 \Rightarrow 11-r-2r = -7 \Rightarrow r=6$ ;  $T_7 = {}^{11}C_6 (a)^5 \left(-\frac{1}{b}\right)^6 = \frac{462a^5}{b^6}$



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**Example: 12** If the coefficients of second, third and fourth term in the expansion of  $(1+x)^{2n}$  are in A.P., then  $2n^2 - 9n + 7$  is equal to

[AMU 2001]

- (a) -1 (b) 0 (c) 1 (d) 3/2

**Solution:** (b)  $T_2 = {}^{2n}C_1, T_3 = {}^{2n}C_2, T_4 = {}^{2n}C_3$  are in A.P. then,  $2 \cdot {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$

$$2 \cdot \frac{2n(2n-1)}{2 \cdot 1} = \frac{2n}{1} + \frac{2n(2n-1)(2n-2)}{3 \cdot 2 \cdot 1}$$

On solving,  $2n^2 - 9n + 7 = 0$

**Example: 13** The coefficient of  $x^5$  in the expansion of  $(1+x^2)^5(1+x)^4$  is

[EAMCET 1996;

UPSEAT 2001; Pb. CET 2002]

- (a) 30 (b) 60 (c) 40 (d) None of these

**Solution:** (b) We have  $(1+x^2)^5(1+x)^4 = ({}^5C_0 + {}^5C_1x^2 + {}^5C_2x^4 + \dots) ({}^4C_0 + {}^4C_1x^1 + {}^4C_2x^2 + \dots)$

So coefficient of  $x^5$  in  $[(1+x^2)^5(1+x)^4] = {}^5C_2 \cdot {}^4C_1 + {}^4C_3 \cdot {}^5C_1 = 60$

**Example: 14** If A and B are the coefficient of  $x^n$  in the expansions of  $(1+x)^{2n}$  and  $(1+x)^{2n-1}$  respectively, then [MP PET 1999

- (a)  $A = B$  (b)  $A = 2B$  (c)  $2A = B$  (d) None of these

**Solution:** (b)  $A = \text{coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{(2n)!}{n!n!} = \frac{2 \cdot (2n-1)!}{(n-1)!n!}$  .....(i)

$B = \text{coefficient of } x^n \text{ in } (1+x)^{2n-1} = {}^{2n-1}C_n = \frac{(2n-1)!}{n!(n-1)!}$  .....(ii)

By (i) and (ii) we get,  $A = 2B$

**Example: 15** The coefficient of  $x^n$  in the expansion of  $(1+x)(1-x)^n$  is

- (a)  $(-1)^{n-1}n$  (b)  $(-1)^n(1-n)$  (c)  $(-1)^{n-1}(n-1)^2$  (d)  $(n-1)$

**Solution:** (b) Coefficient of  $x^n$  in  $(1+x)(1-x)^n = \text{Coefficient of } x^n \text{ in } (1-x)^n + \text{coefficient of } x^{n-1} \text{ in } (1-x)^n$   
 $= \text{Coefficient of } x^n \text{ in } [{}^nC_n(-x)^n + x \cdot {}^nC_{n-1}(-x)^{n-1}] = (-1)^n \cdot {}^nC_n + (-1)^{n-1} \cdot {}^nC_{n-1} = (-1)^n + (-1)^{n-1} \cdot (-n) = (-1)^n [1-n]$ .

### 6.1.6 Number of Terms in the Expansion of $(a+b+c)^n$ and $(a+b+c+d)^n$

$(a+b+c)^n$  can be expanded as :  $(a+b+c)^n = \{(a+b)+c\}^n$   
 $= (a+b)^n + {}^nC_1(a+b)^{n-1}(c)^1 + {}^nC_2(a+b)^{n-2}(c)^2 + \dots + {}^nC_n c^n = (n+1)\text{term} + n\text{term} + (n-1)\text{term} + \dots + 1\text{term}$   
 $\therefore \text{Total number of terms} = (n+1) + (n) + (n-1) + \dots + 1 = \frac{(n+1)(n+2)}{2}$ .

Similarly, Number of terms in the expansion of  $(a+b+c+d)^n = \frac{(n+1)(n+2)(n+3)}{6}$ .

**Example: 16** If the number of terms in the expansion of  $(x-2y+3z)^n$  is 45, then  $n =$

- (a) 7 (b) 8 (c) 9 (d) None of these

**Solution:** (b) Given, total number of terms =  $\frac{(n+1)(n+2)}{2} = 45 \Rightarrow (n+1)(n+2) = 90 \Rightarrow n = 8$ .

**Example: 17** The number of terms in the expansion of  $[(x+3y)^2(3x-y)^2]^3$  is

[Rajasthan PET 1986]

- (a) 14 (b) 28 (c) 32 (d) 56

**Solution:** (b) We have  $[(x+3y)(3x-y)]^6 = [3x^2 + 8xy - 3y^2]^6$ ; Number of terms =  $\frac{(6+1)(6+2)}{2} = 28$

### 6.1.7 Middle Term

The middle term depends upon the value of  $n$ .

(1) When  $n$  is even, then total number of terms in the expansion of  $(x+y)^n$  is  $n+1$  (odd). So there is only one middle term i.e.,  $\left(\frac{n}{2}+1\right)^{\text{th}}$  term is the middle term.  $T_{\left[\frac{n}{2}+1\right]} = {}^n C_{n/2} x^{n/2} y^{n/2}$

(2) When  $n$  is odd, then total number of terms in the expansion of  $(x+y)^n$  is  $n+1$  (even). So, there are two middle terms i.e.,  $\left(\frac{n+1}{2}\right)^{\text{th}}$  and  $\left(\frac{n+3}{2}\right)^{\text{th}}$  are two middle terms.  $T_{\left(\frac{n+1}{2}\right)} = {}^n C_{\frac{n-1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}}$

and  $T_{\left(\frac{n+3}{2}\right)} = {}^n C_{\frac{n+1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}$

**Note** :  $\square$  When there are two middle terms in the expansion then their binomial coefficients are equal.

$\square$  Binomial coefficient of middle term is the greatest binomial coefficient.

**Example: 18** The middle term in the expansion of  $\left(x + \frac{1}{x}\right)^{10}$  is [BIT Ranchi 1991; Rajasthan PET 2002; Pb. CET 1991]

- (a)  ${}^{10}C_4 \frac{1}{x}$  (b)  ${}^{10}C_5$  (c)  ${}^{10}C_5 x$  (d)  ${}^{10}C_7 x^4$

**Solution:** (b)  $\because n$  is even so middle term  $T_{\left(\frac{10}{2}+1\right)} = T_6 \Rightarrow T_6 = T_{5+1} = {}^{10}C_5 x^5 \cdot \frac{1}{x^5} = {}^{10}C_5$

**Example: 19** The middle term in the expansion of  $(1+x)^{2n}$  is [Pb. CET 1998]

- (a)  $\frac{1.3.5\dots(2n-1)}{n!} x^{2n+1}$  (b)  $\frac{2.4.6\dots 2n}{n!} x^{2n+1}$  (c)  $\frac{1.3.5\dots(2n-1)}{n!} x^n$  (d)  $\frac{1.3.5\dots(2n-1)}{n!} x^n \cdot 2^n$

**Solution:** (d) Since  $2n$  is even, so middle term  $= T_{\frac{2n}{2}+1} = T_{n+1} \Rightarrow T_{n+1} = {}^{2n}C_n x^n = \frac{(2n)!}{n!.n!} x^n = \frac{1.3.5\dots(2n-1)}{n!} \cdot 2^n x^n$ .

### 6.1.8 To Determine a Particular Term in the Expansion

In the expansion of  $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$ , if  $x^m$  occurs in  $T_{r+1}$ , then  $r$  is given by  $n\alpha - r(\alpha + \beta) = m \Rightarrow r = \frac{n\alpha - m}{\alpha + \beta}$

Thus in above expansion if constant term which is independent of  $x$ , occurs in  $T_{r+1}$  then  $r$  is determined by

$$n\alpha - r(\alpha + \beta) = 0 \Rightarrow r = \frac{n\alpha}{\alpha + \beta}$$

**Example: 20** The term independent of  $x$  in the expansion of  $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$  is

- (a)  $7/12$  (b)  $7/18$  (c)  $-7/12$  (d)  $-7/16$

**Solution:** (b)  $n=9, \alpha=2, \beta=1$ . Then  $r = \frac{9(2)}{1+2} = 6$ . Hence,  $T_7 = {}^9C_6 \left(\frac{3}{2}\right)^3 \left(-\frac{1}{3}\right)^6 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \cdot \frac{1}{2^3 \cdot 3^3} = \frac{7}{18}$ .

**Example: 21** If the coefficient of  $x^7$  in  $\left(ax^2 + \frac{1}{bx}\right)^{11}$  is equal to the coefficient of  $x^{-7}$  in  $\left(ax - \frac{1}{bx^2}\right)^{11}$  then  $ab =$

[MP PET 1999; AMU 2001; Pb. CET 2002]

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(a) 1

(b)  $1/2$

(c) 2

(d) 3

**Solution:** (a) For coefficient of  $x^7$  in  $\left(ax^2 + \frac{1}{bx}\right)^{11}$ ;  $n = 11, \alpha = 2, \beta = 1, m = 7$

$$r = \frac{11 \cdot 2 - 7}{2 + 1} = \frac{15}{3} = 5$$

Coefficient of  $x^7$  in  $T_6 = {}^{11}C_5 a^6 \cdot \frac{1}{b^5}$  .....(i)

and for coefficient of  $x^{-7}$  in  $\left(ax - \frac{1}{bx^2}\right)^{11}$ ;  $n = 11, \alpha = 1, \beta = 2, m = -7$ ;  $r = \frac{11 \cdot 1 + 7}{3} = 6$

Coefficient of  $x^{-7}$  in  $T_7 = {}^{11}C_6 a^5 \cdot \frac{1}{b^6}$  .....(ii)

From equation (i) and (ii), we get  $ab = 1$

### 6.1.9 Greatest Term and Greatest Coefficient

(1) **Greatest term :** If  $T_r$  and  $T_{r+1}$  be the  $r^{\text{th}}$  and  $(r+1)^{\text{th}}$  terms in the expansion of  $(1+x)^n$ , then

$$\frac{T_{r+1}}{T_r} = \frac{{}^n C_r x^r}{{}^n C_{r-1} x^{r-1}} = \frac{n-r+1}{r} x$$

Let numerically,  $T_{r+1}$  be the greatest term in the above expansion. Then  $T_{r+1} \geq T_r$  or  $\frac{T_{r+1}}{T_r} \geq 1$

$$\therefore \frac{n-r+1}{r} |x| \geq 1 \quad \text{or} \quad r \leq \frac{(n+1)}{(1+|x|)} |x| \quad \text{.....(i)}$$

Now substituting values of  $n$  and  $x$  in (i), we get  $r \leq m + f$  or  $r \leq m$

where  $m$  is a positive integer and  $f$  is a fraction such that  $0 < f < 1$ .

When  $n$  is even  $T_{m+1}$  is the greatest term, when  $n$  is odd  $T_m$  and  $T_{m+1}$  are the greatest terms and both are equal.

**Short cut method :** To find the greatest term (numerically) in the expansion of  $(1+x)^n$ .

(i) Calculate  $m = \left\lfloor \frac{x(n+1)}{x+1} \right\rfloor$

(ii) If  $m$  is integer, then  $T_m$  and  $T_{m+1}$  are equal and both are greatest term.

(iii) If  $m$  is not integer, there  $T_{[m]+1}$  is the greatest term, where  $[.]$  denotes the greatest integral part.

(2) **Greatest coefficient**

(i) If  $n$  is even, then greatest coefficient is  ${}^n C_{n/2}$

(ii) If  $n$  is odd, then greatest coefficient are  ${}^n C_{\frac{n+1}{2}}$  and  ${}^n C_{\frac{n+3}{2}}$

### Important Tips

☞ For finding the greatest term in the expansion of  $(x+y)^n$ . we rewrite the expansion in this form

$$(x+y)^n = x^n \left[1 + \frac{y}{x}\right]^n.$$





Greatest term in  $(x + y)^n = x^n$ . Greatest term in  $\left(1 + \frac{y}{x}\right)^n$

**Example: 22** The largest term in the expansion of  $(3 + 2x)^{50}$ , where  $x = \frac{1}{5}$  is

- (a) 5<sup>th</sup> (b) 8<sup>th</sup> (c) 7<sup>th</sup> (d) 6<sup>th</sup>

**Solution:** (c,d)  $(3 + 2x)^{50} = 3^{50} \left[1 + \frac{2x}{3}\right]^{50}$ , Now greatest term in  $\left(1 + \frac{2x}{3}\right)^{50}$

$$r = \left\lfloor \frac{x(n+1)}{1+x} \right\rfloor = \left\lfloor \frac{\frac{2x}{3}(50+1)}{\frac{2x}{3}+1} \right\rfloor = \left\lfloor \frac{2 \cdot \frac{1}{5} (51)}{\frac{2}{5}+1} \right\rfloor = \left\lfloor \frac{\frac{2}{5} \cdot 51}{\frac{7}{5}} \right\rfloor = \left\lfloor \frac{2 \cdot 51}{7} \right\rfloor = \left\lfloor \frac{102}{7} \right\rfloor = 14 \text{ (an integer)}$$

$\therefore T_r$  and  $T_{[r+1]} = T_6$  and  $T_{[6]+1} = T_7$  are numerically greatest terms

**Example: 23** The greatest coefficient in the expansion of  $(1 + x)^{2n+2}$  is

- (a)  $\frac{(2n)!}{n!^2}$  (b)  $\frac{(2n+2)!}{[(n+1)!]^2}$  (c)  $\frac{(2n+2)!}{n!(n+1)!}$  (d)  $\frac{(2n)!}{n!(n+1)!}$

**Solution:** (b)  $\therefore n$  is even so greatest coefficient in  $(1 + x)^{2n+2}$  is  $= {}^{2n+2}C_{n+1} = \frac{(2n+2)!}{[(n+1)!]^2}$

**Example: 24** The interval in which  $x$  must lie so that the greatest term in the expansion of  $(1 + x)^{2n}$  has the greatest coefficient is

- (a)  $\left(\frac{n-1}{n}, \frac{n}{n-1}\right)$  (b)  $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$  (c)  $\left(\frac{n}{n+2}, \frac{n+2}{n}\right)$  (d) None of these

**Solution:** (b) Here the greatest coefficient is  ${}^{2n}C_n$

$\therefore {}^{2n}C_n x^n > {}^{2n}C_{n+1} x^{n+1} \Rightarrow x > \frac{n}{n+1}$  and  ${}^{2n}C_n x^n > {}^{2n}C_{n-1} x^{n-1} \Rightarrow x < \frac{n+1}{n}$ . Hence the result is (b)

### 6.1.10 Properties of Binomial Coefficients

In the binomial expansion of  $(1 + x)^n$ ,  $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$ .

where  ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$  are the coefficients of various powers of  $x$  and called binomial coefficients, and they are written as  $C_0, C_1, C_2, \dots, C_n$ .

Hence,  $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_r x^r + \dots + C_n x^n$  .....(i)

(1) The sum of binomial coefficients in the expansion of  $(1 + x)^n$  is  $2^n$ .

Putting  $x = 1$  in (i), we get  $2^n = C_0 + C_1 + C_2 + \dots + C_n$  .....(ii)

(2) Sum of binomial coefficients with alternate signs : Putting  $x = -1$  in (i)

We get,  $0 = C_0 - C_1 + C_2 - C_3 + \dots$  .....(iii)

(3) Sum of the coefficients of the odd terms in the expansion of  $(1 + x)^n$  is equal to sum of the coefficients of even terms and each is equal to  $2^{n-1}$ .

From (iii), we have  $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots$  .....(iv)

i.e., sum of coefficients of even and odd terms are equal.

From (ii) and (iv),  $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$  .....(v)

(4)  ${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} {}^{n-2}C_{r-2}$  and so on.



(5) Sum of product of coefficients : Replacing  $x$  by  $\frac{1}{x}$  in (i) we get

$$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n} + \dots \quad \text{(vi)}$$

Multiplying (i) by (vi), we get  $\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \dots) \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots\right)$

Now comparing coefficient of  $x^r$  on both sides. We get,

$${}^{2n}C_{n+r} = C_0C_r + C_1C_{r+1} + \dots + C_{n-r}C_n \quad \dots\text{(vii)}$$

(6) Sum of squares of coefficients : Putting  $r = 0$  in (vii), we get  ${}^{2n}C_n = C_0^2 + C_1^2 + \dots + C_n^2$

$$(7) {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

**Example: 25** The value of  $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots$  is equal to [Karnataka CET 2000]

- (a)  $\frac{2^n - 1}{n + 1}$                       (b)  $n \cdot 2^n$                       (c)  $\frac{2^n}{n}$                       (d)  $\frac{2^n + 1}{n + 1}$

**Solution:** (a) We have  $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots = \frac{n}{2 \cdot 1} + \frac{n(n-1)(n-2)}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{n(n-1)(n-2)(n-3)(n-4)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \dots$   
 $= \frac{1}{n+1} \left[ \frac{(n+1)n}{2!} + \frac{(n+1)n(n-1)(n-2)}{4!} + \dots \right] = \frac{1}{n+1} [2^{(n+1)-1} - 1] = \frac{2^n - 1}{n+1}$

**Trick:** For  $n = 1$ ,  $= \frac{C_1}{2} = \frac{{}^1C_1}{2} = \frac{1}{2}$

Which is given by option (a)  $\frac{2^n - 1}{n + 1} = \frac{2^1 - 1}{1 + 1} = \frac{1}{2}$ .

**Example: 26** The value of  $C_0 + 3C_1 + 5C_2 + \dots + (2n+1)C_n$  is equal to

- (a)  $2^n$                       (b)  $2^n + n \cdot 2^{n-1}$                       (c)  $2^n(n+1)$                       (d) None of these

**Solution:** (c) We have  $C_0 + 3C_1 + 5C_2 + \dots + (2n+1)C_n = \sum_{r=0}^n (2r+1)C_r = \sum_{r=0}^n (2r+1)^n C_r = \sum_{r=0}^n 2r^n C_r + \sum_{r=0}^n C_r$   
 $= 2 \cdot \sum_{r=1}^n r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^n C_r = 2n \sum_{r=1}^n {}^{n-1}C_{r-1} + \sum_{r=0}^n C_r = 2n[(1+1)^{n-1}] + [1+1]^n = 2n \cdot 2^{n-1} + 2^n = 2^n \cdot [n+1]$ .

**Trick:** Put  $n = 1$  in given expansion  ${}^1C_0 + 3 \cdot {}^1C_1 = 1 + 3 = 4$ .

Which is given by option (c)  $2^n \cdot (n+1) = 2^1(1+1) = 4$ .

**Example: 27** If  $S_n = \sum_{r=0}^n \frac{1}{{}^nC_r}$  and  $t_n = \sum_{r=0}^n \frac{r}{{}^nC_r}$ . Then  $\frac{t_n}{S_n}$  is equal to

- (a)  $\frac{2n-1}{2}$                       (b)  $\frac{1}{2}n-1$                       (c)  $n-1$                       (d)  $\frac{n}{2}$

**Solution:** (d) Take  $n = 2m$ , then,  $S_n = \frac{1}{2^m C_0} + \frac{1}{2^m C_1} + \dots + \frac{1}{2^m C_{2m}} = 2 \left[ \frac{1}{2^m C_0} + \frac{1}{2^m C_1} + \dots + \frac{1}{2^m C_{m-1}} \right] + \frac{1}{2^m C_m}$

$$t_n = \sum_{r=0}^n \frac{r}{{}^nC_r} = \sum_{r=0}^{2m} \frac{r}{2^m C_r} = \frac{1}{2^m C_1} + \frac{2}{2^m C_2} + \dots + \frac{2m}{2^m C_{2m}}$$

$$\begin{aligned}
 t_n &= \left( \frac{1}{2^m C_1} + \frac{2m-1}{2^m C_{2m-1}} \right) + \left( \frac{2}{2^m C_2} + \frac{2m-2}{2^m C_{2m-2}} \right) + \dots + \left( \frac{m-1}{2^m C_{m-1}} + \frac{m+1}{2^m C_{m+1}} \right) + \frac{m}{2^m C_m} + \frac{2m}{2^m C_{2m}} \\
 &= 2m \left[ \frac{1}{2^m C_1} + \frac{1}{2^m C_2} + \dots \right] + \frac{m}{2^m C_m} + 2m = 2m \left[ \frac{1}{2^m C_0} + \frac{1}{2^m C_1} + \dots + \frac{1}{2^m C_{m-1}} \right] + \frac{m}{2^m C_m} = m \left[ S_n - \frac{1}{2^m C_m} \right] + \frac{m}{2^m C_m} = m S_n \\
 t_n = m S_n &\Rightarrow \frac{t_n}{S_n} = m = \frac{n}{2}
 \end{aligned}$$

**Example: 28** If  $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$ . Then  $a_0 + a_2 + a_4 + \dots + a_{2n} =$   
**[MNR 1992; DCE 1996; AMU 1998; Rajasthan PET 1999; Karnataka CET 1999; UPSEAT 1999]**

- (a)  $\frac{3^n + 1}{2}$  (b)  $\frac{3^n - 1}{2}$  (c)  $\frac{1 - 3^n}{2}$  (d)  $3^n + \frac{1}{2}$

**Solution: (a)**  $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$   
 Putting  $x = 1$ , we get  $(1-1+1)^n = a_0 + a_1 + a_2 + \dots + a_{2n}$ ;  $1 = a_0 + a_1 + a_2 + \dots + a_{2n}$  .....(i)  
 Again putting  $x = -1$ , we get  $3^n = a_0 - a_1 + a_2 - \dots + a_{2n}$  .....(ii)  
 Adding (i) and (ii), we get,  $3^n + 1 = 2[a_0 + a_2 + a_4 + \dots + a_{2n}]$   
 $\frac{3^n + 1}{2} = a_0 + a_2 + a_4 + \dots + a_{2n}$

**Example: 29** If  $(1+x)^n = \sum_{r=0}^n C_r x^r$ , then  $\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right)\dots\left(1 + \frac{C_n}{C_{n-1}}\right) =$

- (a)  $\frac{n^{n-1}}{(n-1)!}$  (b)  $\frac{(n+1)^{n-1}}{(n-1)!}$  (c)  $\frac{(n+1)^n}{n!}$  (d)  $\frac{(n+1)^{n+1}}{n!}$

**Solution: (c)** We have  $\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right)\dots\left(1 + \frac{C_n}{C_{n-1}}\right) = \left(1 + \frac{n}{1}\right)\left(1 + \frac{n(n-1)/2!}{n}\right)\dots\left(1 + \frac{1}{n}\right)$   
 $= \left(\frac{1+n}{1}\right)\left(\frac{1+n}{2}\right)\left(\frac{1+n}{3}\right)\dots\left(\frac{1+n}{n}\right) = \frac{(n+1)^n}{n!}$

**Trick :** Put  $n = 1, 2, 3, \dots$ ,  $S_1 = 1 + \frac{{}^1C_1}{{}^1C_0} = 2, S_2 = \left(1 + \frac{{}^2C_1}{{}^2C_0}\right)\left(1 + \frac{{}^2C_2}{{}^2C_1}\right) = \frac{9}{2}$

Which is given by option (c)  $n = 1, \frac{(1+1)^1}{1!} = 2$ ; For  $n = 2, \frac{(2+1)^2}{2!} = \frac{9}{2}$

**Example: 30** In the expansion of  $(1+x)^5$ , the sum of the coefficient of the terms is **[Rajasthan PET 1992, 97; Kurukshetra CEE 1992]**  
 (a) 80 (b) 16 (c) 32 (d) 64

**Solution: (c)** Putting  $x = 1$  in  $(1+x)^5$ , the required sum of coefficient =  $(1+1)^5 = 2^5 = 32$

**Example: 31** If the sum of coefficient in the expansion of  $(\alpha^2x^2 - 2\alpha x + 1)^{51}$  vanishes, then the value of  $\alpha$  is **[IIT 1991; Pb. CET 1988]**

- (a) 2 (b) -1 (c) 1 (d) -2

**Solution: (c)** The sum of coefficient of polynomial  $(\alpha^2x^2 - 2\alpha x + 1)^{51}$  is obtained by putting  $x = 1$  in  $(\alpha^2x^2 - 2\alpha x + 1)^{51}$ .  
 Therefore by hypothesis  $(\alpha^2 - 2\alpha + 1)^{51} = 0 \Rightarrow \alpha = 1$

**Example: 32** If  $C_r$  stands for  ${}^nC_r$ , the sum of given series  $\frac{2(n/2)!(n/2)!}{n!} [C_0^2 - 2C_1^2 + 3C_2^2 - \dots + (-1)^n(n+1)C_n^2]$  where  $n$  is an even positive integer, is

- (a) 0 (b)  $(-1)^{n/2}(n+1)$  (c)  $(-1)^n(n+2)$  (d)  $(-1)^{n/2}(n+2)$

**Solution: (d)** We have  $C_0^2 - 2C_1^2 + 3C_2^2 - \dots + (-1)^n(n+1)C_n^2 = [C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^n C_n^2] - [C_1^2 - 2C_2^2 + 3C_3^2 - \dots + (-1)^n n C_n^2]$   
 $= (-1)^{n/2} \cdot n C_{n/2} - (-1)^{n/2-1} \cdot \frac{1}{2} n \cdot n C_{n/2} = (-1)^{n/2} \left[ 1 + \frac{n}{2} \right] n C_{n/2}$

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Therefore the value of given expression =  $\frac{2 \cdot \frac{n!}{2} \cdot \frac{n!}{2}}{n!} \left[ (-1)^{n/2} \cdot \left( 1 + \frac{n}{2} \right) \frac{n!}{\frac{n!}{2} \cdot \frac{n!}{2}} \right] = (-1)^{n/2} (n+2)$

**Example: 33** If  $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ , then the value of  $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n$  will be

[MP PET 1996; Rajasthan PET 1997; DCE 1995; IIT 1971; AMU 1995; EAMCET 2001]

- (a)  $(n+2)2^{n-1}$  (b)  $(n+1)2^n$  (c)  $(n+1)2^{n-1}$  (d)  $(n+2)2^n$

**Solution:** (a) **Trick:** Put  $n=1$  the expansion is equivalent to  ${}^1C_0 + 2 \cdot {}^1C_1 = 1 + 2 = 3$ .

Which is given by option (a) =  $(n+2)2^{n-1} = (1+2)2^0 = 3$

(1) **Use of Differentiation :** This method applied only when the numericals occur as the product of binomial coefficients.

**Solution process :** (i) If last term of the series leaving the plus or minus sign be  $m$ , then divide  $m$  by  $n$  if  $q$  be the quotient and  $r$  be the remainder. *i.e.*,  $m = nq + r$

Then replace  $x$  by  $x^q$  in the given series and multiplying both sides of expansion by  $x^r$ .

(ii) After process (i), differentiate both sides, w.r.t.  $x$  and put  $x=1$  or  $-1$  or  $i$  or  $-i$  etc. according to given series.

(iii) If product of two numericals (or square of numericals) or three numericals (or cube of numerical) then differentiate twice or thrice.

**Example: 34**  $C_1 + 2C_2 + 3C_3 + \dots + nC_n =$

[Rajasthan PET 1995; MP PET 2002]

- (a)  $2^n$  (b)  $n \cdot 2^n$  (c)  $n \cdot 2^{n-1}$  (d)  $n \cdot 2^{n+1}$

**Solution:** (c) We know that,  $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$  .....(i)

Differentiating both sides w.r.t.  $x$ , we get  $n(1+x)^{n-1} = 0 + C_1 + 2 \cdot C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}$

Putting  $x=1$ , we get,  $n \cdot 2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots + nC_n$ .

**Example: 35** If  $n$  is an integer greater than 1, then  $a^{-n}C_1(a-1) + {}^nC_2(a-2) - \dots + (-1)^n(a-n) =$

- (a)  $a$  (b)  $0$  (c)  $a^2$  (d)  $2^n$

**Solution:** (b) We have  $a[C_0 - C_1 + C_2 - \dots] + [C_1 - 2C_2 + 3C_3 - \dots] = a[C_0 - C_1 + C_2 - \dots] - [-C_1 + 2C_2 - 3C_3 + \dots]$

We know that  $(1-x)^n = C_0 - C_1x + C_2x^2 - \dots + (-1)^n C_nx^n$ ; Put  $x=1$ ,  $0 = C_0 - C_1 + C_2 - \dots$

Then differentiating both sides w.r.t. to  $x$ , we get  $n(1-x)^{n-1} = 0 - C_1 + 2C_2x - 3C_3x^2 + \dots$

Put  $x=1$ ,  $0 = -C_1 + 2C_2 - 3C_3 + \dots = a[0] - [0] = 0$ .

(2) **Use of Integration :** This method is applied only when the numericals occur as the denominator of the binomial coefficients.

**Solution process :** If  $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ , then we integrate both sides between the suitable limits which gives the required series.

(i) If the sum contains  $C_0, C_1, C_2, \dots, C_n$  with all positive signs, then integrate between limit 0 to 1.

(ii) If the sum contains alternate signs (*i.e.* +, -) then integrate between limit -1 to 0.

(iii) If the sum contains odd coefficients (*i.e.*,  $C_0, C_2, C_4, \dots$ ) then integrate between -1 to 1.

(iv) If the sum contains even coefficients (*i.e.*,  $C_1, C_3, C_5, \dots$ ) then subtracting (ii) from (i) and then dividing by 2.

(v) If in denominator of binomial coefficients is product of two numericals then integrate two times, first taking limit between 0 to  $x$  and second time take suitable limits.

**Example: 36**  $\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} =$  [Rajasthan PET 1996]

- (a)  $\frac{2^n}{n+1}$                       (b)  $\frac{2^n - 1}{n+1}$                       (c)  $\frac{2^{n+1} - 1}{n+1}$                       (d) None of these

**Solution:** (c) Consider the expansion  $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$  .....(i)  
Integrating both sides of (i) within limits 0 to 1 we get,

$$\int_0^1 (1+x)^n dx = \int_0^1 C_0 + \int_0^1 C_1x + \int_0^1 C_2x^2 + \dots + \int_0^1 C_nx^n dx$$

$$\left[ \frac{(1+x)^{n+1}}{n+1} \right]_0^1 = C_0[x]_0^1 + C_1 \left[ \frac{x^2}{2} \right]_0^1 + \dots + C_n \left[ \frac{x^{n+1}}{n+1} \right]_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = C_0[1] + C_1 \frac{1}{2} + C_2 \frac{1}{3} + \dots + C_n \cdot \frac{1}{n+1}; \quad C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}.$$

**Example: 37**  $2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \dots + \frac{2^{11}}{11}C_{10} =$  [MP PET 1999; EAMCET 1992]

- (a)  $\frac{3^{11} - 1}{11}$                       (b)  $\frac{2^{11} - 1}{11}$                       (c)  $\frac{11^3 - 1}{11}$                       (d)  $\frac{11^2 - 1}{11}$

**Solution:** (a) It is clear that it is a expansion of  $(1+x)^{10} = C_0 + C_1x + C_2x^2 + \dots + C_{10}x^{10}$

Integrating w.r.t.  $x$  both sides between the limit 0 to 2.

$$\left[ \frac{(1+x)^{11}}{11} \right]_0^2 = C_0[x]_0^2 + C_1 \left[ \frac{x^2}{2} \right]_0^2 + C_2 \left[ \frac{x^3}{3} \right]_0^2 + \dots + C_{10} \left[ \frac{x^{11}}{11} \right]_0^2 \Rightarrow \frac{3^{11} - 1}{11} = 2C_0 + \frac{2^2}{2} \cdot C_1 + \frac{2^3}{3} \cdot C_2 + \dots + \frac{2^{11}}{11} C_{10}.$$

**Example: 38** The sum to  $(n+1)$  terms of the following series  $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \dots$  is

- (a)  $\frac{1}{n+1}$                       (b)  $\frac{1}{n+2}$                       (c)  $\frac{1}{n(n+1)}$                       (d) None of these

**Solution:** (d)  $(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \dots$

$$\Rightarrow x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 + \dots \Rightarrow \int_0^1 x(1-x)^n dx = C_0 \left[ \frac{x^2}{2} \right]_0^1 - C_1 \left[ \frac{x^3}{3} \right]_0^1 + C_2 \left[ \frac{x^4}{4} \right]_0^1 - \dots \quad (i)$$

$$\text{The integral on L.H.S. of (i)} = \int_1^0 (1-t)t^n (-dt) \text{ by putting } 1-x=t, \Rightarrow \int_0^1 (t^n - t^{n+1}) dt = \frac{1}{n+1} - \frac{1}{n+2}$$

Whereas the integral on the R.H.S. of (i)

$$= C_0 \left[ \frac{1}{2} \right] - C_1 \left[ \frac{1}{3} \right] + \frac{C_2}{4} - \dots = \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \dots \text{ to } (n+1) \text{ terms} = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$$

**Trick :** Put  $n=1$  in given series  $= \frac{1}{2}C_0 - \frac{1}{3}C_1 = \frac{1}{6}$ . Which is given by option (d).

### 6.1.11 An Important Theorem

If  $(\sqrt{A} + B)^n = I + f$  where  $I$  and  $n$  are positive integers,  $n$  being odd and  $0 \leq f < 1$  then  $(I+f) \cdot f = K^n$  where  $A - B^2 = K > 0$  and  $\sqrt{A} - B < 1$ .

**Note:**  $\square$  If  $n$  is even integer then  $(\sqrt{A} + B)^n + (\sqrt{A} - B)^n = I + f + f'$

Hence L.H.S. and  $I$  are integers.

$\therefore f + f'$  is also integer;  $\Rightarrow f + f' = 1$ ;  $\therefore f' = (1 - f)$



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Hence  $(I + f)(1 - f) = (I + f)f' = (\sqrt{A} + B)^n (\sqrt{A} - B)^n = (A - B^2)^n = K^n$ .

**Example: 39** Let  $R = (5\sqrt{5} + 11)^{2n+1}$  and  $f = R - [R]$  where  $[.]$  denotes the greatest integer function. The value of R.f is [IIT 19

- (a)  $4^{2n+1}$  (b)  $4^{2n}$  (c)  $4^{2n-1}$  (d)  $4^{-2n}$

**Solution:** (a) Since  $f = R - [R]$ ,  $R = f + [R]$

$[5\sqrt{5} + 11]^{2n+1} = f + [R]$ , where  $[R]$  is integer

Now let  $f' = [5\sqrt{5} - 11]^{2n+1}$ ,  $0 < f' < 1$

$$f + [R] - f' = [5\sqrt{5} + 11]^{2n+1} - [5\sqrt{5} - 11]^{2n+1} = 2 \left[ {}^{2n+1}C_1 (5\sqrt{5})^{2n} (11)^1 + {}^{2n+1}C_3 (5\sqrt{5})^{2n-2} (11)^3 + \dots \right]$$

$$= 2 \cdot (\text{Integer}) = 2K \quad (K \in \mathbb{N}) = \text{Even integer}$$

Hence  $f - f' = \text{even integer} - [R]$ , but  $-1 < f - f' < 1$ . Therefore,  $f - f' = 0 \therefore f = f'$

Hence R.f =  $R \cdot f' = (5\sqrt{5} + 11)^{2n+1} (5\sqrt{5} - 11)^{2n+1} = 4^{2n+1}$ .

### 6.1.12 Multinomial Theorem (For positive integral index)

If  $n$  is positive integer and  $a_1, a_2, a_3, \dots, a_m \in \mathbb{C}$  then

$$(a_1 + a_2 + a_3 + \dots + a_m)^n = \sum \frac{n!}{n_1! n_2! n_3! \dots n_m!} a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$$

Where  $n_1, n_2, n_3, \dots, n_m$  are all non-negative integers subject to the condition,  $n_1 + n_2 + n_3 + \dots + n_m = n$ .

(1) The coefficient of  $a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$  in the expansion of  $(a_1 + a_2 + a_3 + \dots + a_m)^n$  is  $\frac{n!}{n_1! n_2! n_3! \dots n_m!}$

(2) The greatest coefficient in the expansion of  $(a_1 + a_2 + a_3 + \dots + a_m)^n$  is  $\frac{n!}{(q!)^{m-r} [(q+1)!]^r}$

Where  $q$  is the quotient and  $r$  is the remainder when  $n$  is divided by  $m$ .

(3) If  $n$  is +ve integer and  $a_1, a_2, \dots, a_m \in \mathbb{C}$ ,  $a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$  then coefficient of  $x^r$  in the expansion of  $(a_1 + a_2 x + \dots + a_m x^{m-1})^n$  is  $\sum \frac{n!}{n_1! n_2! n_3! \dots n_m!}$

Where  $n_1, n_2, \dots, n_m$  are all non-negative integers subject to the condition:  $n_1 + n_2 + \dots + n_m = n$  and  $n_2 + 2n_3 + 3n_4 + \dots + (m-1)n_m = r$ .

(4) The number of distinct or dissimilar terms in the multinomial expansion  $(a_1 + a_2 + a_3 + \dots + a_m)^n$  is  ${}^{n+m-1}C_{m-1}$ .

**Example: 40** The coefficient of  $x^5$  in the expansion of  $(x^2 - x - 2)^5$  is

- (a) - 83 (b) - 82 (c) - 81 (d) 0

**Solution:** (c) Coefficient of  $x^5$  in the expansion of  $(x^2 - x - 2)^5$  is  $\sum \frac{5!}{n_1! n_2! n_3!} (1)^{n_1} (-1)^{n_2} (-2)^{n_3}$ .

where  $n_1 + n_2 + n_3 = 5$  and  $2n_1 - n_2 - 2n_3 = 5$ . The possible value of  $n_1, n_2$  and  $n_3$  are shown in margin

$n_1$	$n_2$	$n_3$
1	3	1
2	1	2
0	5	0

$$\therefore \text{The coefficient of } x^5 = \frac{5!}{1!3!1!} (1)^1 (-1)^3 (-2)^1 + \frac{5!}{2!1!2!} (1)^2 (-1)^1 (-2)^2 + \frac{5!}{0!5!0!} (1)^0 (-1)^5 (-2)^0 = 40 - 120 - 1 = -81$$



**Example: 41** Find the coefficient of  $a^3b^4c^5$  in the expansion of  $(bc + ca + ab)^6$   
 (a) 0 (b) 60 (c) -60 (d) None of these

**Solution:** (b) In this case,  $a^3b^4c^5 = (ab)^x(bc)^y(ca)^z = a^{x+z}.b^{x+y}.c^{y+z}$   
 $z + x = 3, x + y = 4, y + z = 5; 2(x + y + z) = 12; x + y + z = 6$ . Then  $x = 1, y = 3, z = 2$

Therefore the coefficient of  $a^3b^4c^5$  in the expansion of  $(bc + ca + ab)^6 = \frac{6!}{1!3!2!} = 60$ .

### 6.1.13 Binomial Theorem for any Index

**Statement :**  $(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots$  terms up to  $\infty$

When  $n$  is a negative integer or a fraction, where  $-1 < x < 1$ , otherwise expansion will not be possible.

If  $x < 1$ , the terms of the above expansion go on decreasing and if  $x$  be very small a stage may be reached when we may neglect the terms containing higher power of  $x$  in the expansion, then  $(1 + x)^n = 1 + nx$ .

#### Important Tips

☞ Expansion is valid only when  $-1 < x < 1$ .

☞  ${}^nC_r$  can not be used because it is defined only for natural number, so  ${}^nC_r$  will be written as  $\frac{n(n-1)\dots(n-r+1)}{r!}$

☞ The number of terms in the series is infinite.

☞ If first term is not 1, then make first term unity in the following way,  $(x + y)^n = x^n \left[ 1 + \frac{y}{x} \right]^n$ , if  $\left| \frac{y}{x} \right| < 1$ .

**General term :**  $T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$

**Some important expansions:**

(i)  $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$

(ii)  $(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}(-x)^r + \dots$

(iii)  $(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}x^r + \dots$

(iv)  $(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}(-x)^r + \dots$

(a) **Replace  $n$  by 1 in (iii) :**  $(1 - x)^{-1} = 1 + x + x^2 + \dots + x^r + \dots \infty$ , General term,  $T_{r+1} = x^r$

(b) **Replace  $n$  by 1 in (iv) :**  $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-x)^r + \dots \infty$ , General term,  $T_{r+1} = (-x)^r$ .

(c) **Replace  $n$  by 2 in (iii) :**  $(1 - x)^{-2} = 1 + 2x + 3x^2 + \dots + (r+1)x^r + \dots \infty$ , General term,  $T_{r+1} = (r+1)x^r$ .

(d) **Replace  $n$  by 2 in (iv) :**  $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots + (r+1)(-x)^r + \dots \infty$   
 General term,  $T_{r+1} = (r+1)(-x)^r$ .

(e) **Replace  $n$  by 3 in (iii) :**  $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(r+1)(r+2)}{2!}x^r + \dots \infty$

General term,  $T_{r+1} = (r+1)(r+2)/2!.x^r$

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(f) **Replace  $n$  by 3 in (iv) :**  $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + \frac{(r+1)(r+2)}{2!}(-x)^r + \dots \infty$

General term,  $T_{r+1} = \frac{(r+1)(r+2)}{2!}(-x)^r$

**Example: 42** To expand  $(1+2x)^{-1/2}$  as an infinite series, the range of  $x$  should be **[AMU 2002]**

- (a)  $\left[-\frac{1}{2}, \frac{1}{2}\right]$                       (b)  $\left(-\frac{1}{2}, \frac{1}{2}\right)$                       (c)  $[-2, 2]$                       (d)  $(-2, 2)$

**Solution:** (b)  $(1+2x)^{-1/2}$  can be expanded if  $|2x| < 1$  i.e., if  $|x| < \frac{1}{2}$  i.e., if  $-\frac{1}{2} < x < \frac{1}{2}$  i.e., if  $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ .

**Example: 43** If the value of  $x$  is so small that  $x^2$  and higher power can be neglected, then  $\frac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{1+x + \sqrt{1+x}}$  is equal to **[Roorkee 1962]**

- (a)  $1 + \frac{5}{6}x$                       (b)  $1 - \frac{5}{6}x$                       (c)  $1 + \frac{2}{3}x$                       (d)  $1 - \frac{2}{3}x$

**Solution:** (b) Given expression can be written as  $= \frac{(1+x)^{1/2} + (1-x)^{2/3}}{1+x + (1+x)^{1/2}} =$

$$\frac{\left(1 + \frac{1}{2}x + \left(-\frac{1}{8}\right)x^2 + \dots\right) + \left(1 - \frac{2}{3}x - \frac{1}{9}x^2 - \dots\right)}{1+x + \left[1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots\right]}$$

$$= \frac{1 - \frac{1}{12}x - \frac{1}{144}x^2 + \dots}{1 + \frac{3}{4}x - \frac{1}{16}x^2 + \dots} = 1 - \frac{5}{6}x + \dots = 1 - \frac{5}{6}x, \text{ when } x^2, x^3 \dots \text{ are neglected.}$$

**Example: 44** If  $(1+ax)^n = 1 + 8x + 24x^2 + \dots$  then the value of  $a$  and  $n$  is (d) 1, 2

(a) 2, 4                      (b) 2, 3                      (c) 3, 6

**Solution:** (a) We know that  $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \dots$

$$(1+ax)^n = 1 + \frac{n(ax)}{1!} + \frac{n(n-1)(ax)^2}{2!} + \dots \Rightarrow 1 + 8x + 24x^2 + \dots = 1 + \frac{n(ax)}{1!} + \frac{n(n-1)(ax)^2}{2!} + \dots$$

Comparing coefficients of both sides we get,  $na = 8$ , and  $\frac{n(n-1)a^2}{2!} = 24$  on solving,  $a = 2, b = 4$ .

**Example: 45** Coefficient of  $x^r$  in the expansion of  $(1-2x)^{-1/2}$  **[Kurukshetra CEE 2001]**

- (a)  $\frac{(2r)!}{(r!)^2}$                       (b)  $\frac{(2r)!}{2^r \cdot (r!)^2}$                       (c)  $\frac{(2r)!}{(r!)^2 \cdot 2^{2r}}$                       (d)  $\frac{(2r)!}{2^r \cdot (r+1)!(r-1)!}$

**Solution:** (b) Coefficient of

$$x^r = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\dots\left(-\frac{1}{2}-r+1\right)}{r!}(-2)^r = \frac{1.3.5\dots(2r-1) \cdot (-1)^r \cdot (-1)^r \cdot 2^r}{2^r r!} = \frac{1.3.5\dots(2r-1)}{r!} = \frac{(2r)!}{r!r!2^r}$$

**Example: 46** The coefficient of  $x^{25}$  in  $(1+x+x^2+x^3+x^4)^{-1}$  is (d) - 1

(a) 25                      (b) - 25                      (c) 1

**Solution:** (c) Coefficient of  $x^{25}$  in  $(1+x+x^2+x^3+x^4)^{-1}$



$$\begin{aligned}
&= \text{Coefficient of } x^{25} \text{ in } \left[ \frac{1(1-x^5)}{1-x} \right]^{-1} = \text{Coefficient of } x^{25} \text{ in } (1-x^5)^{-1} \cdot (1-x) \\
&= \text{Coefficient of } x^{25} \text{ in } [(1-x^5)^{-1} - x(1-x^5)^{-1}] = [1 + (x^5)^1 + (x^5)^2 + \dots] - x[1 + (x^5)^1 + (x^5)^2 + \dots] \\
&= \text{Coefficient of } x^{25} \text{ in } [1 + x^5 + x^{10} + x^{15} + \dots] - \text{Coefficient of } x^{24} \text{ in } [1 + x^5 + x^{10} + x^{15} + \dots] = 1 - 0 = 1.
\end{aligned}$$

**Example: 47**  $1 - \frac{1}{8} + \frac{1}{8} \cdot \frac{3}{16} - \dots =$  [EAMCET 1990]

(a)  $\frac{2}{5}$                       (b)  $\frac{\sqrt{2}}{5}$                       (c)  $\frac{2}{\sqrt{5}}$                       (d) None of these

**Solution:** (c) We know that  $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \infty$

Here  $nx = -\frac{1}{8}, \frac{n(n-1)}{2}x^2 = \frac{3}{8 \cdot 16} \Rightarrow x = \frac{1}{4}, n = -\frac{1}{2} \Rightarrow 1 - \frac{1}{8} + \frac{1}{8} \cdot \frac{3}{16} - \dots = \left(1 + \frac{1}{4}\right)^{-1/2} = \frac{2}{\sqrt{5}}$ .

**Example: 48** If  $x$  is so small that its two and higher power can be neglected and  $(1-2x)^{-1/2}(1-4x)^{-5/2} = 1+kx$  then  $k =$

[Rajasthan PET 1993]

(a) 1                      (b) -2                      (c) 10                      (d) 11

**Solution:** (d)  $(1-2x)^{-1/2}(1-4x)^{-5/2} = 1+kx$

$$\left[ 1 + \frac{(-1/2)(-2x)}{1!} + \frac{(-1/2)(-3/2)(-2x)^2}{2!} + \dots \right] \left[ 1 + \frac{(-5/2)(-4x)}{1!} + \frac{(-5/2)(-7/2)(-4x)^2}{2!} + \dots \right] = 1+kx$$

Higher power can be neglected. Then  $\left[1 + \frac{x}{1!}\right] \left[1 + \frac{10x}{1!}\right] = 1+kx$ ;  $1+10x+x = 1+kx$ ;  $k = 11$

**Example: 49** The cube root of  $1+3x+6x^2+10x^3+\dots$  is

(a)  $1-x+x^2-x^3+\dots \infty$     (b)  $1+x^3+x^6+x^9+\dots$     (c)  $1+x+x^2+x^3+\dots$     (d) None of these

**Solution:** (c) We have  $(1+3x+6x^2+10x^3+\dots)^{1/3} = [(1-x)^{-3}]^{1/3}$ ; [ $\because (1-x)^{-3} = 1+3x+6x^2+\dots \infty$ ]  $\Rightarrow$   
 $(1-x)^{-1} = 1+x+x^2+\dots \infty$

**Example: 50** The coefficient of  $x^n$  in the expansion of  $\left(\frac{1}{1-x}\right)\left(\frac{1}{3-x}\right)$  is

(a)  $\frac{3^{n+1}-1}{2 \cdot 3^{n+1}}$                       (b)  $\frac{3^{n+1}-1}{3^{n+1}}$                       (c)  $2\left(\frac{3^{n+1}-1}{3^{n+1}}\right)$                       (d) None of these

**Solution:** (a)  $\frac{1}{(1-x)(3-x)} = (1-x)^{-1}(3-x)^{-1} = 3^{-1}(1-x)^{-1}\left(1-\frac{x}{3}\right)^{-1} = \frac{1}{3}[1+x+x^2+\dots \infty] \left[1+\frac{x}{3}+\frac{x^2}{3^2}+\dots+\frac{x^{n-1}}{3^{n-1}}+\frac{x^n}{3^n}\right]$

Coefficient of  $x^n = \frac{1}{3^{n+1}} + \frac{1}{3^n} + \frac{1}{3^{n-1}} + \dots + (n+1)$  terms  $= \frac{1}{3^{n+1}} \frac{[3^{n+1}-1]}{3-1} = \frac{3^{n+1}-1}{2 \cdot 3^{n+1}}$ .

**Trick:** Put  $n=1, 2, 3, \dots$  and find the coefficients of  $x, x^2, x^3, \dots$  and comparing with the given option as

Coefficient of  $x^2$  is  $= \frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3^1} = \frac{1}{3^3} \frac{[3^3-1]}{3-1} = \frac{13}{27}$ ; Which is given by option (a)

$$\frac{3^{n+1}-1}{2 \cdot (3^{n+1})} = \frac{3^3-1}{2 \cdot 3^3} = \frac{13}{27}$$

### 6.1.14 Three / Four Consecutive terms or Coefficients

(1) **If consecutive coefficients are given:** In this case divide consecutive coefficients pair wise. We get equations and then solve them.

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(2) **If consecutive terms are given :** In this case divide consecutive terms pair wise i.e. if four consecutive terms be  $T_r, T_{r+1}, T_{r+2}, T_{r+3}$  then find  $\frac{T_r}{T_{r+1}}, \frac{T_{r+1}}{T_{r+2}}, \frac{T_{r+2}}{T_{r+3}} \Rightarrow \lambda_1, \lambda_2, \lambda_3$  (say) then divide  $\lambda_1$  by  $\lambda_2$  and  $\lambda_2$  by  $\lambda_3$  and solve.

**Example: 51** If  $a_1, a_2, a_3, a_4$  are the coefficients of any four consecutive terms in the expansion of  $(1+x)^n$ , then

$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} =$$

[IIT 1975]

(a)  $\frac{a_2}{a_2 + a_3}$       (b)  $\frac{1}{2} \frac{a_2}{a_2 + a_3}$       (c)  $\frac{2a_2}{a_2 + a_3}$       (d)  $\frac{2a_3}{a_2 + a_3}$

**Solution:** (c) Let  $a_1, a_2, a_3, a_4$  be respectively the coefficients of  $(r+1)^{\text{th}}, (r+2)^{\text{th}}, (r+3)^{\text{th}}, (r+4)^{\text{th}}$  terms in the expansion of  $(1+x)^n$ . Then  $a_1 = {}^n C_r, a_2 = {}^n C_{r+1}, a_3 = {}^n C_{r+2}, a_4 = {}^n C_{r+3}$ .

$$\begin{aligned} \text{Now, } \frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} &= \frac{{}^n C_r}{{}^n C_r + {}^n C_{r+1}} + \frac{{}^n C_{r+2}}{{}^n C_{r+2} + {}^n C_{r+3}} = \frac{{}^n C_r}{{}^{n+1} C_{r+1}} + \frac{{}^n C_{r+2}}{{}^{n+1} C_{r+3}} = \frac{{}^n C_r}{r+1} + \frac{{}^n C_{r+2}}{r+3} \\ &= \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2(r+2)}{n+1} = 2 \cdot \frac{{}^n C_{r+1}}{{}^{n+1} C_{r+2}} = 2 \cdot \frac{{}^n C_{r+1}}{{}^n C_{r+1} + {}^n C_{r+2}} = \frac{2a_2}{a_2 + a_3} \end{aligned}$$

### 6.1.15 Some Important Points

#### (1) Pascal's Triangle :

1		$(x+y)^0$				
1	1	$(x+y)^1$				
1	2	1	$(x+y)^2$			
1	3	3	1	$(x+y)^3$		
1	4	6	4	1	$(x+y)^4$	
1	5	10	10	5	1	$(x+y)^5$

Pascal's triangle gives the direct binomial coefficients.

**Example :**  $(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

#### (2) Method for finding terms free from radical or rational terms in the expansion of

$(a^{1/p} + b^{1/q})^N \nexists a, b \in \text{prime numbers}$  : Find the general term  $T_{r+1} = {}^N C_r (a^{1/p})^{N-r} (b^{1/q})^r = {}^N C_r a^{\frac{N-r}{p}} b^{\frac{r}{q}}$

Putting the values of  $0 \leq r \leq N$ , when indices of  $a$  and  $b$  are integers.

**Note :**  $\square$  Number of irrational terms = Total terms - Number of rational terms.

**Example: 52** The number of integral terms in the expansion of  $(\sqrt{3} + \sqrt[8]{5})^{256}$  is

[AIEEE 2003]

(a) 32      (b) 33      (c) 34      (d) 35

**Solution:** (b)  $T_{r+1} = {}^{256} C_r \cdot 3^{\frac{256-r}{2}} \cdot 5^{\frac{r}{8}}$

First term =  ${}^{256} C_0 \cdot 3^{128} \cdot 5^0 = \text{integer}$  and after eight terms, i.e., 9<sup>th</sup> term =  ${}^{256} C_8 \cdot 3^{124} \cdot 5^1 = \text{integer}$

Continuing like this, we get an A.P., 1<sup>st</sup>, 9<sup>th</sup>, ..... 257<sup>th</sup>;  $T_n = a + (n-1)d \Rightarrow 257 = 1 + (n-1)8 \Rightarrow n = 33$

**Example: 53** The number of irrational terms in the expansion of  $(\sqrt[8]{5} + \sqrt[6]{2})^{100}$  is

- (a) 97                                      (b) 98                                      (c) 96                                      (d) 99

**Solution:** (a)  $T_{r+1} = {}^{100}C_r 5^{\frac{100-r}{8}} \cdot 2^{\frac{r}{6}}$

As 2 and 5 are co-prime.  $T_{r+1}$  will be rational if  $100 - r$  is multiple of 8 and  $r$  is multiple of 6 also

$$0 \leq r \leq 100$$

$$\therefore r = 0, 6, 12, \dots, 96; \therefore 100 - r = 4, 10, 16, \dots, 100 \quad \dots(i)$$

But  $100 - r$  is to be multiple of 8.

$$\text{So, } 100 - r = 0, 8, 16, 24, \dots, 96 \quad \dots(ii)$$

Common terms in (i) and (ii) are 16, 40, 64, 88.

$\therefore r = 84, 60, 36, 12$  give rational terms  $\therefore$  The number of irrational terms =  $101 - 4 = 97$ .

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## 6.2 Mathematical Induction

### 6.2.1 First Principle of Mathematical Induction

The proof of proposition by mathematical induction consists of the following three steps :

**Step I :** (Verification step) : Actual verification of the proposition for the starting value “ $i$ ”

**Step II :** (Induction step) : Assuming the proposition to be true for “ $k$ ”,  $k \geq i$  and proving that it is true for the value  $(k + 1)$  which is next higher integer.

**Step III :** (Generalization step) : To combine the above two steps

Let  $p(n)$  be a statement involving the natural number  $n$  such that

(i)  $p(1)$  is true i.e.  $p(n)$  is true for  $n = 1$ .

(ii)  $p(m + 1)$  is true, whenever  $p(m)$  is true i.e.  $p(m)$  is true  $\Rightarrow p(m + 1)$  is true.

Then  $p(n)$  is true for all natural numbers  $n$ .

### 6.2.2 Second Principle of Mathematical Induction

The proof of proposition by mathematical induction consists of following steps :

**Step I :** (Verification step) : Actual verification of the proposition for the starting value  $i$  and  $(i + 1)$ .

**Step II :** (Induction step) : Assuming the proposition to be true for  $k - 1$  and  $k$  and then proving that it is true for the value  $k + 1$ ;  $k \geq i + 1$ .

**Step III :** (Generalization step) : Combining the above two steps.

Let  $p(n)$  be a statement involving the natural number  $n$  such that

(i)  $p(1)$  is true i.e.  $p(n)$  is true for  $n = 1$  and

(ii)  $p(m + 1)$  is true, whenever  $p(n)$  is true for all  $n$ , where  $i \leq n \leq m$

Then  $p(n)$  is true for all natural numbers.

For  $a \neq b$ , The expression  $a^n - b^n$  is divisible by

(a)  $a + b$  if  $n$  is even.

(b)  $a - b$  if  $n$  is odd or even.

### 6.2.3 Some Formulae based on Principle of Induction

For any natural number  $n$

$$(i) \sum n = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} \quad (ii)$$

$$\sum n^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(iii) \sum n^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4} = (\sum n)^2$$

**Example: 1** The smallest positive integer  $n$ , for which  $n! < \left(\frac{n+1}{2}\right)^n$  hold is

- (a) 1 (b) 2 (c) 3 (d) 4

**Solution:** (b) Let  $P(n) : n! < \left(\frac{n+1}{2}\right)^n$

**Step I :** For  $n = 2 \Rightarrow 2! < \left(\frac{2+1}{2}\right)^2 \Rightarrow 2 < \frac{9}{4} \Rightarrow 2 < 2.25$  which is true. Therefore,  $P(2)$  is true.

**Step II :** Assume that  $P(k)$  is true, then  $p(k) : k! < \left(\frac{k+1}{2}\right)^k$

**Step III :** For  $n = k + 1$ ,

$$\begin{aligned}
 P(k+1) : (k+1)! &< \left(\frac{k+2}{2}\right)^{k+1} \Rightarrow k! < \left(\frac{k+1}{2}\right)^k \Rightarrow (k+1)k! < \frac{(k+1)^{k+1}}{2^k} \\
 \Rightarrow (k+1)! &< \frac{(k+1)^{k+1}}{2^k} \quad \dots(i) \quad \text{and} \quad \frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1} \quad \dots(ii) \\
 \Rightarrow \left(\frac{k+2}{k+1}\right)^{k+1} &> 2 \Rightarrow \left[1 + \frac{1}{k+1}\right]^{k+1} > 2 \Rightarrow 1 + (k+1)\frac{1}{k+1} + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + \dots > 2 \\
 \Rightarrow 1 + 1 + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 &+ \dots > 2
 \end{aligned}$$

Which is true, hence (ii) is true.

From (i) and (ii),  $(k+1)! < \frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1} \Rightarrow (k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$

Hence  $P(k+1)$  is true. Hence by the principle of mathematical induction  $P(n)$  is true for all  $n \in N$

**Trick :** By check option

(a) For  $n = 1, 1! < \left(\frac{1+1}{2}\right)^1 \Rightarrow 1 < 1$  which is wrong (b) For  $n = 2, 2! < \left(\frac{3}{2}\right)^2 \Rightarrow 2 < \frac{9}{4}$  which is correct

(c) For  $n = 3, 3! < \left(\frac{3+1}{2}\right)^3 \Rightarrow 6 < 8$  which is correct

(d) For  $n = 4, 4! < \left(\frac{4+1}{2}\right)^4 \Rightarrow 24 < \left(\frac{5}{2}\right)^4 \Rightarrow 24 < 39.0625$  which is correct.

But smallest positive integer  $n$  is 2.

**Example: 2** Let  $S(k) = 1 + 3 + 5 + \dots + (2k - 1) = 3 + k^2$ . Then which of the following is true.

[AIEEE 2004]

- (a) Principle of mathematical induction can be used to prove the formula  
 (b)  $S(k) \Rightarrow S(k+1)$   
 (c)  $S(k) \not\Rightarrow S(k+1)$   
 (d)  $S(1)$  is correct

**Solution:** (c) We have  $S(k) = 1 + 3 + 5 + \dots + (2k - 1) = 3 + k^2$ ,  $S(1) \Rightarrow 1 = 4$ , Which is not true and  $S(2) \Rightarrow 3 = 7$ , Which is not true.

Hence induction cannot be applied and  $S(k) \not\Rightarrow S(k+1)$

**Example: 3** When  $P$  is a natural number, then  $P^{n+1} + (P+1)^{2n-1}$  is divisible by

[IIT 1994]

- (a)  $P$  (b)  $P^2 + P$  (c)  $P^2 + P + 1$  (d)  $P^2 - 1$

**Solution:** (c) For  $n = 1$ , we get,  $P^{n+1} + (P+1)^{2n-1} = P^2 + (P+1)^1 = P^2 + P + 1$ ,

Which is divisible by  $P^2 + P + 1$ , so result is true for  $n = 1$

Let us assume that the given result is true for  $n = m \in N$

i.e.  $P^{m+1} + (P+1)^{2m-1}$  is divisible by  $P^2 + P + 1$  i.e.  $P^{m+1} + (P+1)^{2m-1} = k(P^2 + P + 1) \quad \forall k \in N \quad \dots(i)$

Now,  $P^{(m+1)+1} + (P+1)^{2(m+1)-1} = P^{m+2} + (P+1)^{2m+1} = P^{m+2} + (P+1)^2(P+1)^{2m-1}$

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$$\begin{aligned}
 &= P^{m+2} + (P+1)^2 [k(P^2 + P + 1) - P^{m+1}] && \text{by using (i)} \\
 &= P^{m+2} + (P+1)^2 \cdot k(P^2 + P + 1) - (P+1)^2 (P)^{m+1} = P^{m+1} [P - (P+1)^2] + (P+1)^2 \cdot k(P^2 + P + 1) \\
 &= P^{m+1} [P - P^2 - 2P - 1] + (P+1)^2 \cdot k(P^2 + P + 1) = -P^{m+1} [P^2 + P + 1] + (P+1)^2 \cdot k(P^2 + P + 1) \\
 &= (P^2 + P + 1) [k \cdot (P+1)^2 - P^{m+1}]
 \end{aligned}$$

Which is divisible by  $P^2 + P + 1$ , so the result is true for  $n = m + 1$ . Therefore, the given result is true for all  $n \in N$  by induction.

**Trick :** For  $n = 2$ , we get,  $P^{n+1} + (P+1)^{2n-1} = P^3 + (P+1)^3 = P^3 + P^3 + 1 + 3P^2 + 3P = 2P^3 + 3P^2 + 3P + 1$

Which is divisible by  $P^2 + P + 1$ . Given result is true for all  $n \in N$

**Example: 4** Given  $U_{n+1} = 3U_n - 2U_{n-1}$  and  $U_0 = 2, U_1 = 3$ , the value of  $U_n$  for all  $n \in N$  is

- (a)  $2^n - 1$                       (b)  $2^n + 1$                       (c) 0                      (d) None of these

**Solution:** (b)  $\because U_{n+1} = 3U_n - 2U_{n-1}$                       .....(i)

**Step I :** Given  $U_1 = 3$

For  $n = 1$ ,  $U_{1+1} = 3U_1 - 2U_0, U_2 = 3 \cdot 3 - 2 \cdot 2 = 5$

Option (b)  $U_n = 2^n + 1$

For  $n = 1, U_1 = 2^1 + 1 = 3$  which is true. For  $n = 2, U_2 = 2^2 + 1 = 5$  which is true

Therefore, the result is true for  $n = 1$  and  $n = 2$

**Step II :** Assume it is true for  $n = k$  then it is also true for  $n = k - 1$

$$\text{Then } U_k = 2^k + 1 \quad \text{.....(ii) and } U_{k-1} = 2^{k-1} + 1 \quad \text{.....(iii)}$$

**Step III :** Putting  $n = k$  in (i), we get

$$U_{k+1} = 3U_k - 2U_{k-1} = 3[2^k + 1] - 2[2^{k-1} + 1] = 3 \cdot 2^k + 3 - 2 \cdot 2^{k-1} - 2 = 3 \cdot 2^k + 1 - 2 \cdot 2^{k-1}$$

$$\Rightarrow 3 \cdot 2^k - 2^k + 1 = 2 \cdot 2^k + 1 = 2^{k+1} + 1 \Rightarrow U_{k+1} = 2^{k+1} + 1$$

This shows that the result is true for  $n = k + 1$ , by the principle of mathematical induction the result is true for all  $n \in N$ .

### 6.2.4 Divisibility Problems

To show that an expression is divisible by an integer

(i) If  $a, p, n, r$  are positive integers, then first of all we write  $a^{pn+r} = a^{pn} \cdot a^r = (a^p)^n \cdot a^r$ .

(ii) If we have to show that the given expression is divisible by  $c$ .S

Then express,  $a^p = [1 + (a^p - 1)]$ , if some power of  $(a^p - 1)$  has  $c$  as a factor.

$a^p = [2 + (a^p - 2)]$ , if some power of  $(a^p - 2)$  has  $c$  as a factor.

$a^p = [K + (a^p - K)]$ , if some power of  $(a^p - K)$  has  $c$  as a factor.

**Example: 5**  $(1+x)^n - nx - 1$  is divisible by (where  $n \in N$ )

- (a)  $2x$                       (b)  $x^2$                       (c)  $2x^3$                       (d) All of these

**Solution:** (b)  $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \Rightarrow (1+x)^n - nx - 1 = x^2 \left[ \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}x + \dots \right]$

From above it is clear that  $(1+x)^n - nx - 1$  is divisible by  $x^2$ .

**Trick :**  $(1+x)^n - nx - 1$ . Put  $n = 2$  and  $x = 3$ ; Then  $4^2 - 2 \cdot 3 - 1 = 9$

Is not divisible by 6, 54 but divisible by 9. Which is given by option (c) =  $x^2 = 9$ .

**Example: 6** The greatest integer which divides the number  $101^{100} - 1$  is

(a) 100

(b) 1000

(c) 10000

(d) 100000

**Solution:** (c)  $(1 + 100)^{100} = 1 + 100 \cdot 100 + \frac{100 \cdot 99}{1 \cdot 2} (100)^2 + \dots \Rightarrow 101^{100} - 1 = 100 \cdot 100 \left[ 1 + \frac{100 \cdot 99}{1 \cdot 2} + \frac{100 \cdot 99 \cdot 98}{3 \cdot 2 \cdot 1} 100 + \dots \right]$

From above it is clear that,  $101^{100} - 1$  is divisible by  $(100)^2 = 10000$

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